

# A relation of cusp forms and Maass forms on product of hyperbolic Riemann orbisurfaces of finite volume

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## Abstract

In [2], J. Jorgenson and J. Kramer proved a certain key identity which relates the two natural metrics, namely the hyperbolic metric and the canonical metric defined on a compact hyperbolic Riemann surface. In this article, we extend this identity to product of noncompact hyperbolic Riemann orbisurfaces of finite volume, which can be realized as a quotient space of the action of a Fuchsian subgroup of first kind on the hyperbolic upper half plane.

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## 1 Introduction

For  $i = 1, 2$ , let  $X_i$  be a noncompact hyperbolic Riemann orbisurface of finite volume  $\text{vol}_{\text{hyp}}(X_i)$  with genus  $g_i \geq 1$ , and can be realized as the quotient space  $\Gamma_i \backslash \mathbb{H}$ , where  $\Gamma_i \subset \text{PSL}_2(\mathbb{R})$  is a Fuchsian subgroup of the first kind acting on the hyperbolic upper half-plane  $\mathbb{H}$ , via fractional linear transformations. Let  $\mathcal{E}_i$  and  $\mathcal{P}_i$  denote the finite sets of elliptic fixed points and cusps of  $\Gamma_i$ , respectively. Put  $\overline{X}_i = X_i \cup \mathcal{P}_i$ . Then,  $\overline{X}_i$  admits the structure of a Riemann surface. Now consider the complex surface  $X = X_1 \times X_2$ , which admits the structure of a noncompact Kähler-orbifold of dimension two. Put  $\overline{X} = \overline{X}_1 \times \overline{X}_2$ .

For  $i = 1, 2$ , let  $\mu_{\text{hyp}}^i$  denote the  $(1,1)$ -form associated to hyperbolic metric, which is the natural metric on  $X_i$ , and of constant negative curvature minus one. Now  $\mu_{\text{hyp}}^1 + \mu_{\text{hyp}}^2$  is the natural metric on  $X$ , and let  $\mu_{\text{hyp}}^{\text{vol}}$  denote the volume form associated to  $\mu_{\text{hyp}}^1 + \mu_{\text{hyp}}^2$ .

For  $i = 1, 2$ , the Riemann surface  $\overline{X}_i$  is embedded in its Jacobian variety  $\text{Jac}(\overline{X}_i)$  via the Abel-Jacobi map. Then, the pull back of the flat Euclidean metric by the Abel-Jacobi map is called the canonical metric, and the  $(1,1)$ -form associated to it is denoted by  $\overline{\mu}_{\text{can}}^i$ . We denote its restriction to  $X_i$  by  $\mu_{\text{can}}^i$ . Now,  $\mu_{\text{can}}^1 + \mu_{\text{can}}^2$  defines a metric on  $X$ , which corresponds to the flat Euclidean metric, and let  $\mu_{\text{can}}^{\text{vol}}$  denote the volume form associated to  $\mu_{\text{can}}^1 + \mu_{\text{can}}^2$ .

For  $i = 1, 2$ , let  $\Delta_{\text{hyp}}^i$  denote the hyperbolic Laplacian acting on smooth functions on  $X_i$ . The hyperbolic heat kernel  $K_{\text{hyp}}^i(t; z_i, w_i)$  on  $\mathbb{R}_{>0} \times X_i \times X_i$  is the unique solution of the heat equation

$$\left( \Delta_{\text{hyp}}^i + \frac{\partial}{\partial t} \right) K_{\text{hyp}}^i(t; z_i, w_i) = 0,$$

with the normalization condition

$$\lim_{t \rightarrow 0} \int_{X_i} K_{\text{hyp}}^i(t; z_i, w_i) f(z_i) \mu_{\text{hyp}}^i(z_i) = f(w_i),$$

for any fixed  $w \in X_i$  and any smooth function  $f$  on  $X_i$ . When  $z_i = w_i$ , for brevity of notation, we denote the hyperbolic heat kernel by  $K_{\text{hyp}}^i(t; z_i)$ .

**Main result** With notation as above, for  $z = (z_1, z_2) \in (X_1 \setminus \mathcal{E}_1) \times (X_2 \setminus \mathcal{E}_2)$ , we have the relation

of differential forms

$$\begin{aligned} g_1 g_2 \mu_{\text{can}}^{\text{vol}}(z) &= \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X_1)} \right) \cdot \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X_2)} \right) \mu_{\text{hyp}}^{\text{vol}}(z) + \frac{1}{2} \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X_1)} \right) \times \\ &\left( \int_0^\infty \Delta_{\text{hyp}}^2 K_{\text{hyp}}^2(t; z_2) dt \right) \mu_{\text{hyp}}^{\text{vol}}(z) + \frac{1}{2} \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X_2)} \right) \cdot \left( \int_0^\infty \Delta_{\text{hyp}}^1 K_{\text{hyp}}^1(t; z_1) dt \right) \mu_{\text{hyp}}^{\text{vol}}(z) + \\ &\frac{1}{4} \left( \int_0^\infty \Delta_{\text{hyp}}^1 K_{\text{hyp}}^1(t; z_1) dt \right) \cdot \left( \int_0^\infty \Delta_{\text{hyp}}^2 K_{\text{hyp}}^2(t; z_2) dt \right) \mu_{\text{hyp}}^{\text{vol}}(z). \end{aligned}$$

The above relation, which we call the key-identity, relates the two natural metrics defined on a Kähler-orbifold of dimension two. The key-identity is proved for compact hyperbolic Riemann surfaces, by J. Jorgenson and J. Kramer in [2]. The same authors extended the key-identity to noncompact hyperbolic Riemann surfaces of finite volume in [3]. Following different methods, the key-identity is extended to noncompact hyperbolic Riemann orbisurfaces in [1].

**Arithmetic significance** The key-identity is the most significant technical result of [2], which transforms a problem in Arakelov theory into that of hyperbolic geometry. The key-identity has enabled J. Jorgenson and J. Kramer to derive optimal bounds for the canonical Green's function defined on a compact hyperbolic Riemann surface  $X$  in terms of invariants coming from the hyperbolic geometry of  $X$ .

Using the key-identity one can relate the holomorphic world of cusp forms with the  $C^\infty$  world of Mäss forms, via the spectral expansion of the hyperbolic heat kernel in terms of Mäss forms. In fact, J. Jorgenson and J. Kramer have derived a Rankin-Selberg  $L$ -function relation relating the Fourier coefficients of cusp forms with those of Mäss forms in [3].

Our main result is the first instance of an extension of the key-identity to higher dimensions, relating the cusp forms of the group  $\Gamma_1 \times \Gamma_2$  with the Mäss forms defined on  $X_1$  and  $X_2$  via the spectral expansions of the hyperbolic heat kernels  $K_{\text{hyp}}^1(t; z_1)$  and  $K_{\text{hyp}}^2(t; z_2)$ , respectively.

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## 2 Background material

For  $i = 1, 2$ , let  $\Gamma_i \subset \text{PSL}_2(\mathbb{R})$  be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane  $\mathbb{H}$ . Let  $X_i$  be the quotient space  $\Gamma_i \backslash \mathbb{H}$ , and let  $g_i$  denote the genus of  $X_i$ . The quotient space  $X_i$  admits the structure of a Riemann orbisurface.

Let  $\mathcal{E}_i$  and  $\mathcal{P}_i$  be the finite sets of elliptic fixed points and cusps of  $X_i$ , respectively; put  $\mathcal{S}_i = \mathcal{E}_i \cup \mathcal{P}_i$ . For  $\mathfrak{e}_i \in \mathcal{E}_i$ , let  $m_{\mathfrak{e}_i}$  denote the order of  $\mathfrak{e}_i$ ; for  $p_i \in \mathcal{P}_i$ , put  $m_{p_i} = \infty$ ; for  $z_i \in X_i \setminus \mathcal{E}_i$ , put  $m_{z_i} = 1$ . Let  $\overline{X}_i$  denote  $\overline{X}_i = X_i \cup \mathcal{P}_i$ .

Locally, away from the elliptic fixed points and the cusps, we identify  $\overline{X}_i$  with its universal cover  $\mathbb{H}$ , and hence, denote the points on  $\overline{X}_i \setminus \mathcal{S}_i$  by the same letter as the points on  $\mathbb{H}$ .

The quotient space  $\overline{X}_i$  admits the structure of a compact Riemann surface. We refer the reader to Section 1.8 in [4], for the details regarding the structure of  $\overline{X}_i$  as a compact Riemann surface.

Let  $X$  denote the product of the Riemann orbisurfaces  $X_1 \times X_2$ . Then,  $X$  admits the structure of a complex Kähler-orbifold of dimension two. The boundary of  $X$  is given by  $\partial X = (X_1 \times \mathcal{P}_2) \cup (X_2 \times \mathcal{P}_1)$ , and the compactification of  $X$  is given by  $\overline{X} = \overline{X}_1 \times \overline{X}_2$ .

**Hyperbolic metric** For  $i = 1, 2$ , we denote the (1,1)-form corresponding to the hyperbolic metric of  $X_i$ , which is compatible with the complex structure on  $X_i$  and has constant negative curvature equal to minus one, by  $\mu_{\text{hyp}}^i(z_i)$ . Locally, for  $z_i \in X_i \setminus \mathcal{E}_i$ , it is given by

$$\mu_{\text{hyp}}^i(z_i) = \frac{i}{2} \cdot \frac{dz_i \wedge d\bar{z}_i}{\text{Im}(z_i)^2}.$$

From the above formula, it follows that the hyperbolic metric  $\mu_{\text{hyp}}^i(z_i)$  is singular at the elliptic fixed points and at the cusps.

Let  $\text{vol}_{\text{hyp}}(X_i)$  be the volume of  $X_i$  with respect to the hyperbolic metric  $\mu_{\text{hyp}}^i(z_i)$ . It is given by the formula

$$\text{vol}_{\text{hyp}}(X_i) = 2\pi \left( 2g_i - 2 + |\mathcal{P}_i| + \sum_{\mathfrak{e}_i \in \mathcal{E}_i} \left( 1 - \frac{1}{m_{\mathfrak{e}_i}} \right) \right).$$

We denote the (1,1)-form corresponding to the hyperbolic metric of  $X$ , which is compatible with the complex structure on  $X$ , by  $\mu_{\text{hyp}}(z)$  and the corresponding volume form by  $\mu_{\text{hyp}}^{\text{vol}}(z)$ . Locally, for  $z = (z_1, z_2) \in (X_1 \setminus \mathcal{E}_1) \times (X_2 \setminus \mathcal{E}_2)$ , it is given by

$$\mu_{\text{hyp}}(z) = \mu_{\text{hyp}}^1(z_1) + \mu_{\text{hyp}}^2(z_2) = \frac{i}{2} \cdot \frac{dz_1 \wedge d\bar{z}_1}{\text{Im}(z_1)^2} + \frac{i}{2} \cdot \frac{dz_2 \wedge d\bar{z}_2}{\text{Im}(z_2)^2},$$

and the corresponding volume form is given by

$$\mu_{\text{hyp}}^{\text{vol}}(z) = \mu_{\text{hyp}}^1(z_1) \wedge \mu_{\text{hyp}}^2(z_2) = -\frac{dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2}{2 \text{Im}(z_1)^2 \text{Im}(z_2)^2}.$$

**Canonical metric** For  $i = 1, 2$ , let  $S_2(\Gamma_i)$  denote the  $\mathbb{C}$ -vector space of cusp forms of weight 2 with respect to  $\Gamma_i$  equipped with the Petersson inner-product. Let  $\{f_1^i, \dots, f_{g_i}^i\}$  denote an orthonormal basis of  $S_2(\Gamma_i)$  with respect to the Petersson inner-product. Then, the (1,1)-form  $\mu_{\text{can}}^i(z_i)$  corresponding to the canonical metric of  $X_i$  is given by

$$\mu_{\text{can}}^i(z_i) = \frac{i}{2g_i} \sum_{j=1}^{g_i} |f_j^i(z_i)|^2 dz_i \wedge d\bar{z}_i.$$

The canonical metric  $\mu_{\text{can}}^i(z)$  remains smooth at the elliptic fixed points and at the cusps, and measures the volume of  $X$  to be one.

Let  $\Omega_{\overline{X}_i}^1$  denote the cotangent bundle of holomorphic differential forms of degree one on  $\overline{X}_i$ . Recall that for each  $f^i \in S_2(\Gamma_i)$ ,  $f^i(z_i)dz_i$  defines a holomorphic differential form of degree one on  $\overline{X}_i$ , and every holomorphic differential form of degree one on  $\overline{X}_i$  comes from a weight 2 cusp form. So  $\{f_1^i, \dots, f_{g_i}^i\}$  the orthonormal basis of  $S_2(\Gamma_i)$  with respect to the Petersson inner-product gives us an orthonormal basis  $\{f_1^i dz_i, \dots, f_{g_i}^i dz_i\}$  of  $H^0(\overline{X}_i, \Omega_{\overline{X}_i}^1)$  endowed with the  $L^2$ -inner product given by

$$\langle \alpha^i, \beta^i \rangle = \frac{i}{2} \int_{\overline{X}} \alpha^i(z_i) \overline{\beta^i(z_i)},$$

where  $\alpha^i, \beta^i \in \Omega_{\overline{X}_i}^1$ .

Let  $\Omega_{\overline{X}}^2$  denote the space of holomorphic differential forms of degree 2, and let  $\{\omega_1, \dots, \omega_n\}$  denote an orthonormal basis of  $H^0(\overline{X}, \Omega_{\overline{X}}^2)$  endowed with the  $L^2$ -inner product given by

$$\langle \alpha, \beta \rangle = -\frac{1}{4} \int_{\overline{X}} \alpha(z) \overline{\beta(z)},$$

where  $n$  denotes the dimension of  $H^0(\overline{X}, \Omega_{\overline{X}}^2)$  as a vector space over  $\mathbb{C}$ , and  $\alpha, \beta \in \Omega_{\overline{X}}^2$ . Then, the canonical volume form on  $X$  is defined as

$$\mu_{\text{can}}^{\text{vol}}(z) = -\frac{1}{4n} \sum_{j=1}^n \omega_j(z) \wedge \overline{\omega_j(z)}.$$

The canonical volume form  $\mu_{\text{can}}^{\text{vol}}(z)$  measures the volume of  $X$  to be one.

**Hyperbolic Laplacian** For  $i = 1, 2$ , the hyperbolic Laplacian acting on smooth functions defined on  $X_i$  is given by

$$\Delta_{\text{hyp}}^i = -y_i^2 \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right),$$

and the hyperbolic Laplacian acting on smooth functions defined on  $X$  is given by

$$\Delta_{\text{hyp}} = \Delta_{\text{hyp}}^1 + \Delta_{\text{hyp}}^2 = -y_1^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right) - y_2^2 \left( \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} \right).$$

**Hyperbolic heat kernels** For  $t \in \mathbb{R}_{>0}$  and  $z, w \in \mathbb{H}$ , let  $K_{\mathbb{H}}(t; z, w)$  denote the hyperbolic heat kernel on  $\mathbb{R}_{>0} \times \mathbb{H} \times \mathbb{H}$ .

For  $i = 1, 2$ ,  $t \in \mathbb{R}_{>0}$  and  $z_i, w_i \in X_i$ , the hyperbolic heat kernel  $K_{\text{hyp}}^i(t; z_i, w_i)$  on  $\mathbb{R}_{>0} \times X_i \times X_i$  is defined as

$$K_{\text{hyp}}^i(t; z_i, w_i) = \sum_{\gamma_i \in \Gamma_i} K_{\mathbb{H}}(t; z_i, \gamma_i w_i). \quad (1)$$

For  $z_i, w_i \in X_i$ , the hyperbolic heat kernel  $K_{\text{hyp}}^i(t; z_i, w_i)$  satisfies the differential equation

$$\left( \Delta_{\text{hyp}, z}^i + \frac{\partial}{\partial t} \right) K_{\text{hyp}}^i(t; z, w) = 0, \quad (2)$$

Furthermore, for a fixed  $w_i \in X_i$ , and any smooth function  $f^i$  on  $X_i$ , the hyperbolic heat kernel  $K_{\text{hyp}}^i(t; z_i, w_i)$  satisfies the equation

$$\lim_{t \rightarrow 0} \int_{X_i} K_{\text{hyp}}^i(t; z_i, w_i) f^i(z) \mu_{\text{hyp}}^i(z_i) = f^i(w_i). \quad (3)$$

To simplify notation, we write  $K_{\text{hyp}}^i(t; z_i)$  instead of  $K_{\text{hyp}}^i(t; z_i, z_i)$ , when  $z_i = w_i$ .

**Key-identity on  $X_i$**  For  $i = 1, 2$ ,  $z_i \in X_i \setminus \mathcal{E}_i$ , we have the relation of differential forms

$$g_i \mu_{\text{can}}^i(z_i) = \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X_i)} \right) \mu_{\text{hyp}}^i(z_i) + \frac{1}{2} \left( \int_0^\infty \Delta_{\text{hyp}}^i K_{\text{hyp}}^i(t; z_i) dt \right) \mu_{\text{hyp}}^i(z_i). \quad (4)$$

This relation has been established as Theorem 3.4 in [2], when  $X_i$  is compact. The proof given in [2] applies to our case where  $X$  does admit elliptic fixed points and cusps, as long as  $z_i \in X_i \setminus \mathcal{E}_i$ .

In [1], the above identity is extended to elliptic fixed points and cusps at the level of currents.

### 3 Key-identity on $X$

**Lemma 1.** *The dimension of  $H^0(\overline{X}, \Omega_{\overline{X}}^2)$  as a vector space over  $\mathbb{C}$  is  $g_1 g_2$ , and for  $z = (z_1, z_2) \in X$ , the canonical volume form is given by*

$$\mu_{\text{can}}^{\text{vol}}(z) = \mu_{\text{can}}^1(z_1) \wedge \mu_{\text{can}}^2(z_2) = -\frac{1}{4g_1 g_2} \sum_{j=1}^{g_1} \sum_{k=1}^{g_2} |f_j^1(z_1)|^2 \cdot |f_k^2(z_2)|^2 dz_1 \wedge d\overline{z}_1 \wedge dz_2 \wedge d\overline{z}_2. \quad (5)$$

*Proof.* From Künneth theorem of algebraic geometry, we have

$$H^0(\overline{X}, \Omega_{\overline{X}}^2) = H^0(\overline{X}_1, \Omega_{\overline{X}_1}^1) \otimes H^0(\overline{X}_2, \Omega_{\overline{X}_2}^1).$$

So from the isomorphism  $S_2(\Gamma_i) \cong H^0(\overline{X}_i, \Omega_{\overline{X}_i}^1)$ , it follows that the set

$$\{f_j^1 f_k^2\}_{\substack{1 \leq j \leq g_1 \\ 1 \leq k \leq g_2}}$$

forms an orthonormal basis of  $H^0(\overline{X}, \Omega_{\overline{X}}^2)$ , which implies that

$$\mu_{\text{can}}^{\text{vol}}(z) = -\frac{1}{4g_1g_2} \sum_{j=1}^{g_1} \sum_{k=1}^{g_2} |f_j^1(z_1)|^2 \cdot |f_k^2(z_2)|^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

Furthermore, a direct calculation, shows that  $\mu_{\text{can}}^1(z_1) \wedge \mu_{\text{can}}^2(z_2) = \mu_{\text{can}}^{\text{vol}}(z)$ , which completes the proof of the lemma.  $\square$

**Theorem 2.** *For  $z = (z_1, z_2) \in (X_1 \setminus \mathcal{E}_1) \times (X_2 \setminus \mathcal{E}_2)$ , we have the relation of differential forms*

$$\begin{aligned} g_1g_2 \mu_{\text{can}}^{\text{vol}}(z) &= \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X_1)} \right) \cdot \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X_2)} \right) \mu_{\text{hyp}}^{\text{vol}}(z) + \frac{1}{2} \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X_1)} \right) \times \\ &\left( \int_0^\infty \Delta_{\text{hyp}}^2 K_{\text{hyp}}^2(t; z_2) dt \right) \mu_{\text{hyp}}^{\text{vol}}(z) + \frac{1}{2} \left( \frac{1}{4\pi} + \frac{1}{\text{vol}_{\text{hyp}}(X_2)} \right) \cdot \left( \int_0^\infty \Delta_{\text{hyp}}^1 K_{\text{hyp}}^1(t; z_1) dt \right) \mu_{\text{hyp}}^{\text{vol}}(z) + \\ &\frac{1}{4} \left( \int_0^\infty \Delta_{\text{hyp}}^1 K_{\text{hyp}}^1(t; z_1) dt \right) \cdot \left( \int_0^\infty \Delta_{\text{hyp}}^2 K_{\text{hyp}}^2(t; z_2) dt \right) \mu_{\text{hyp}}^{\text{vol}}(z). \end{aligned}$$

*Proof.* The proof of the theorem follows from combining equations (4) and (5).  $\square$

## References

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